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INVERSE LAMINAR BOUNDARY-LAYER PROBLEMS WITH ASSIGNED SHEAR. THE MECHUL FUNCTION REVISITED

K. C. KAUFMAN* AND G. H. HOFFMAN†

Applied Research Laboratory, The Pennsylvania State University, Post Office Box 30, State College, PA 16804, U.S.A.

SUMMARY

An inverse method is presented which accurately determines the pressure distribution for assigned wall shear in a two-dimensional, laminar, incompressible boundary layer. The method reformulates the mechul function scheme of Cebeci and Keller to produce a stable solution in the marching direction and to increase accuracy in the normal direction. In the reformulation a modified pressure gradient parameter variation in the normal direction is used in conjunction with three-point backward differences for streamwise derivatives and fourth-order accurate splines for normal derivatives. The resulting spline–finite difference equations are solved by Newton–Raphson iteration together with partial pivoting. Numerical solutions are presented for self-similar and non self-similar flows and compared with published results.

KEY WORDS: Laminar Boundary Layer Splines Incompressible

INTRODUCTION

One method for the solution of the inverse two-dimensional, laminar boundary layer problem with assigned wall shear is the mechul function scheme of Cebeci and Keller.[1] In this method, the over determined property brought about by the extra boundary condition is eliminated by specifying an extra differential equation governing the normal variation of the pressure gradient parameter. As a result, the system is solved as an extension of the standard boundary-layer problem. For the self-similar case the method is very accurate and efficient but in the non self-similar case it suffers from a weak instability. The instability appears as a disturbance that develops in the far field and propagates towards the wall, ultimately destroying the solution as it is marched downstream.

In this study, the mechul function method of Reference 1 is reformulated to eliminate the instability in the non self-similar case and the discretization is changed to increase the accuracy for a given number of mesh points. In the original formulation the Keller box method, which was used to discretize the differential equations, appears to be the culprit for producing the instability. The reason is that owing to its centering in the middle of the mesh cell it is not fully implicit. The new formulation uses three-point backward differences to approximate streamwise derivatives and fourth-order accurate splines for normal derivatives. The fully implicit method leads to a completely stable procedure for the non self-similar case.

The resulting spline–finite difference equations are solved by the Newton–Raphson technique, as in Reference 1. Because splines are used, partial pivoting is necessary to prevent the buildup of

* Graduate Assistant.
† Senior Research Associate.

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roundoff errors during the L–U decomposition solution of the block matrix system at a particular streamwise station. The use of splines also required a slight recasting of the extra differential equation on the pressure gradient parameter.

GOVERNING DIFFERENTIAL EQUATIONS

The governing equations in this problem are those for a two-dimensional, incompressible, laminar boundary layer. Using standard similarity variables the equations reduce to the following single equation for the pseudo self-similar stream function f:

$$f_{\eta\eta\eta} + ff_{\eta\eta} + \beta(1 - f_{\eta}^{2}) = 2\xi(f_{\eta}f_{\xi\eta} - f_{\eta\eta}f_{\xi}), \tag{1}$$

where the subscripts denote partial differentiation with respect to the subscripted variable, ξ is the streamwise variable, η is the similarity variable in the normal direction, β is the pressure gradient parameter defined by

$$\beta(\xi) = \frac{2\xi}{u_{e}}\frac{du_{e}}{d\xi} \tag{2}$$

and u_{e} is the velocity at the edge of the boundary layer. The boundary conditions are as follows:

$$\left.\begin{aligned} f(\xi, 0) = f_{\eta}(\xi, 0) = 0, \\ f_{\eta}(\xi, \eta_{\infty}) = 1 \quad , \end{aligned}\right\} \tag{3}$$

where for simplicity η_{∞}, the location of the outer edge of the boundary layer, is taken as a suitably large constant. Thus the computational domain is the rectangle $0 \leqslant \xi \leqslant \xi_{max}, 0 \leqslant \eta \leqslant \eta_{\infty}$. Equations (1) and (3) define the standard problem.

For the inverse problem, the wall shear is specified as

$$f_{\eta\eta}(\xi, 0) = S(\xi), \quad \xi \geqslant 0. \tag{4}$$

This leads to an over determined system in equations (1), (3) and (4). To complete the formulation of the mechul function scheme, the pressure gradient parameter is written as

$$\beta(\xi) = \beta(\xi, \eta),$$

and the following derivative condition is introduced:[1]

$$\frac{\partial\beta}{\partial\eta} = 0, \quad \xi \geqslant 0. \tag{5}$$

The pressure gradient parameter is thus determined by solving equations (1) and (5) with boundary conditions (3) and (4).

For self-similar flows, $f = f(\eta)$, from which it follows that the pressure gradient parameter and the boundary conditions are also independent of ξ. The governing equations thus reduce to ordinary differential equations, viz.

$$f''' + ff'' + \beta(1 - f'^{2}) = 0, \tag{6}$$

$$\beta' = 0, \tag{7}$$

where the prime denotes differentiation with respect to η.

In the course of the present study, the original derivative condition on β, equation (5), when used with the fourth-order spline relation $S^{1}(4, 0)$, was found to produce a singular matrix in the L–U decomposition solution of the resulting block tridiagonal system (the spline notation used throughout this paper is the same as that in Reference 2). When a constant normal stepsize is used, a singular block matrix occurs on the diagonal during the forward substitution step which cannot be eliminated by rearranging the equations. The singularity is due entirely to the form of the spline relation $S^1(4,0)$.

To eliminate this singularity, the requirement on the normal pressure gradient parameter was changed to

$$\frac{\partial^2\beta}{\partial\eta^2}=0, \quad \xi\geqslant 0. \tag{8}$$

Then, the condition

$$\frac{\partial\beta}{\partial\eta}=0, \quad \xi\geqslant 0 \tag{5}$$

is enforced only at the wall with two-point spline boundary conditions. Using the alternate fourth-order spline relation $S^2(4,0)$ for equation (8) produces a non-singular matrix for constant or varying stepsize while still forcing the pressure gradient parameter to be independent of η. This modification introduces no additional equations on the solution besides the enforcement of equation (5) at the wall.

SPLINE-FINITE DIFFERENCE EQUATIONS

As originally formulated in this study, equation (1) was written in first-order form so that together with equation (5) all η-derivatives could be represented by the spline relation $S^1(4,0)$. In the modified formulation, where equation (8) replaces equation (5), equation (1) is still written in first-order form while equation (8) remains unchanged. Thus the governing system of equations is

$$f_\eta=u, \tag{9a}$$

$$u_\eta=\tau, \tag{9b}$$

$$\beta_{\eta\eta}=0, \tag{9c}$$

$$\tau_\eta=\beta(u^2-1)-f\tau+2\xi(uu_\xi-\tau f_\xi), \tag{9d}$$

and the boundary conditions become

$$\left.\begin{aligned} f(\xi,0)&=0\\ u(\xi,0)&=0\\ \tau(\xi,0)&=S(\xi)\\ u(\xi,\eta_\infty)&=1\end{aligned}\right\}. \tag{10}$$

The ξ-derivatives in equation (9d) are approximated by the following generalized backward difference formula for constant stepsize $\Delta\xi$:

$$(g_\xi)_{i,j}=\frac{1}{\Delta\xi}[ag_{i,j}+bg_{i-1,j}+cg_{i-2,j}]. \tag{11}$$

For interior grid points, the three-point, second-order accurate version of equation (11) is used, for which

$$a=\tfrac{3}{2}, \quad b=-2 \quad \text{and } c=\tfrac{1}{2},$$

whereas for the second downstream station only the two-point version is used where

$$a=1,\quad b=-1\quad \text{and}\quad c=0.$$

Thus applying equation (11) to u_ξ and f_ξ in equation (9d) yields

$$(\tau_\eta)_{i,j}=\beta_{i,j}(u_{i,j}^2-1)-f_{i,j}\tau_{i,j}+\alpha_i(au_{i,j}^2+bu_{i-1,j}^2+cu_{i-2,j}^2)$$
$$-2\alpha_i(af_{i,j}\tau_{i,j}+bf_{i-1,j}\tau_{i,j}+cf_{i-2,j}\tau_{i,j}) \qquad (12)$$

where

$$\alpha_i=\frac{\xi_i}{\Delta\xi_i}.$$

Since splines are to be used in the η-direction, the following spline derivative approximations are introduced:

$$\left.\begin{aligned} l^f&=f_\eta\\ l^u&=u_\eta\\ l^\tau&=\tau_\eta\\ l^\beta&=\beta_\eta\\ L^\beta&=\beta_{\eta\eta}\end{aligned}\right\}. \qquad (13)$$

Then equations (9a) to (9c) and (12) at grid point (i,j) can be written as

$$l_{i,j}^f=u_{i,j}, \qquad (14a)$$

$$l_{i,j}^u=\tau_{i,j}, \qquad (14b)$$

$$L_{i,j}^\beta=0, \qquad (14c)$$

$$l_{i,j}^\tau=\beta_{i,j}(u_{i,j}^2-1)-(2a\alpha_i+1)f_{i,j}\tau_{i,j}-2\alpha_i(bf_{i-1,j}+cf_{i-2,j})\tau_{i,j}$$
$$+\alpha_i(au_{i,j}^2+bu_{i-1,j}^2+cu_{i-2,j}^2). \qquad (14d)$$

Equations (14) contain the eight unknowns f, u, τ, β, l^f, l^u, l^τ and L^β. Hence to close the system four tridiagonal spline relations are needed. We use $S^1(4,0)$ to relate l^f to f, l^u to u and l^τ to τ; then $S^2(4,0)$ to relate L^β to β. The expressions for these splines are[2]

$$S^1(4,0):\quad l_{j+1}^g+(1+\sigma)^2l_j^g+\sigma^2 l_{j-1}^g$$
$$=\frac{2}{1+\sigma}\frac{1}{h_j}\left[\frac{1+2\sigma}{\sigma}g_{j+1}+\frac{\sigma-1}{\sigma}(1+\sigma)^3g_j-\sigma^2(2+\sigma)g_{j-1}\right], \qquad (15)$$

and

$$S^2(4,0):\quad \frac{\sigma^2+\sigma-1}{12\sigma}L_{j+1}^g+\frac{\sigma^3+4\sigma^2+4\sigma+1}{12\sigma}L_j^g+\frac{1+\sigma-\sigma^2}{12}L_{j-1}^g$$
$$=\frac{1}{h_j^2}\left[\frac{g_{j+1}}{\sigma}-\frac{1+\sigma}{\sigma}g_j+g_{j-1}\right], \qquad (16)$$

where

$$h_j=\Delta\eta_j,\quad \sigma=\sigma_j=\frac{h_{j+1}}{h_j}\quad\text{and}\quad l^g=g_\eta.$$

Equations (14a) to (14d) can be used to eliminate l^f, l^u, L^β and l^τ in the four tridiagonal spline relations, thus reducing the number of unknowns at (i,j) to four—f, u, β and τ. The resulting tridiagonal equations are (subscript i understood)

$$\sigma 2u_{j-1} + \sigma 1u_j + u_{j+1} = \hat{\sigma}3f_{j-1} + \hat{\sigma}4f_j + \hat{\sigma}5f_{j+1}, \tag{17a}$$

$$\sigma 2\tau_{j-1} + \sigma 1\tau_j + \tau_{j+1} = \hat{\sigma}3u_{j-1} + \hat{\sigma}4u_j + \hat{\sigma}5u_{j+1}, \tag{17b}$$

$$\hat{\sigma}8\beta_{j-1} + \hat{\sigma}7\beta_j + \hat{\sigma}6\beta_{j+1} = 0, \tag{17c}$$

$$\begin{aligned}&\sigma 2[(c1+\beta)u^2 - \beta + c2f\tau + \bar{E}1_j\tau]_{j-1} + \sigma 1[(c1+\beta)u^2 - \beta + c2f\tau + \bar{E}2_j\tau]_j \\ &+ [(c1+\beta)u^2 - \beta + c2f\tau + \bar{E}3_j\tau]_{j+1} = -\sigma 2c4_{j-1} - \sigma 1c4_j - c4_{j+1}.\end{aligned} \tag{17d}$$

The coefficients in equations (17) can be found in Reference 3.

The block tridiagonal relations given by equation (17) form a non-linear system relating f, u, τ and β. These equations are linearized for solution by the Newton–Raphson technique. If superscript (n) denotes the iteration number associated with the spline–finite difference equations at station i, and F denotes one of the four unknowns f, u, τ or β, then the Newton iterate is given by

$$F^{(n+1)} = F^{(n)} + \delta F^{(n)}, \tag{18}$$

where $\delta F^{(n)}$ is assumed to be a small correction. The linearized system is obtained by substituting equation (18) into equations (17) and neglecting squares of $\delta F^{(n)}$, then writing the result in terms of the corrections. The linearized block tridiagonal correction equations at line i are

$$-\hat{\sigma}3\delta f_{j-1} + \sigma 2\delta u_{j-1} - \hat{\sigma}4\delta f_j + \sigma 1\delta u_j - \hat{\sigma}5\delta f_{j+1} + \delta u_{j+1} = L_j, \tag{19a}$$

$$-\hat{\sigma}3\delta u_{j-1} + \sigma 2\delta\tau_{j-1} - \hat{\sigma}4\delta u_j + \sigma 1\delta\tau_j - \hat{\sigma}5\delta u_{j+1} + \delta\tau_{j+1} = P_j, \tag{19b}$$

$$-\hat{\sigma}8\delta\beta_{j-1} - \hat{\sigma}7\delta\beta_j - \hat{\sigma}6\delta\beta_{j+1} = Q_j \tag{19c}$$

$$\begin{aligned}&\sigma^2[\hat{A}\delta f + \hat{B}\delta u + \hat{D}\delta\tau + \hat{G}\delta\beta]_{j-1} + \sigma 1[\hat{A}\delta f + \hat{B}\delta u + \hat{E}\delta\tau + \hat{G}\delta\beta]_j \\ &+ [\hat{A}\delta f + \hat{B}\delta u + \hat{F}\delta\tau + \hat{G}\delta\beta]_{j+1} = T_j,\end{aligned} \tag{19d}$$

where $\hat{A}_j$, $\hat{B}_j$, $\hat{D}_j$, $\hat{E}_j$, $\hat{F}_j$ and G_j are coefficients obtained from the linearization of the equations (see Appendix A of Reference 3).

These block tridiagonal equations can be written in the following matrix form (i subscript understood):

$$B_jZ_{j-1} + A_jZ_j + C_jZ_{j+1} = R_j \tag{20}$$

for $2 \leqslant j \leqslant N$, where N is the number of intervals in η and

$$Z_j = \begin{bmatrix}\delta f\\ \delta u\\ \delta\tau\\ \delta\beta\end{bmatrix}_j. \tag{21}$$

A_j, B_j, C_j are 4×4 matrices whose elements are obtained from the four correction equations, and R_j is a four-component column vector of known quantities obtained from the right-hand sides of the correction equations.

The boundary conditions at the wall in correction form, at streamwise station i, are

$$\delta f_1 = 0, \tag{22a}$$

$$\delta u_1 = 0, \tag{22b}$$

$$\delta\tau_1 = 0. \tag{22c}$$

One additional condition must be provided to close this system. The two-point spline relation, given in Reference 2, is used:

$$\beta_2 - \beta_1 - \frac{h_2}{2}(l_2^\beta + l_1^\beta) + \frac{h_2^2}{12}(L_2^\beta - L_1^\beta) = 0. \tag{23}$$

This relation allows the original pressure gradient condition to be enforced at the wall. From the governing equation, $L_j^\beta = 0$, and enforcing the original gradient condition $l_j^\beta = 0$, equation (23) reduces to the simple relationship $\beta_2 = \beta_1$, which in correction form is

$$\delta\beta_2 = \delta\beta_1. \tag{24}$$

The linearized block system for the corrections at the wall can then be written in the following matrix form:

$$A_1 Z_1 + C_1 Z_2 = R_1 \tag{25}$$

with Z_j defined by equation (21).

The far field boundary condition at station i is

$$\delta u_{N+1} = 0. \tag{26}$$

Three additional conditions are required to close this system. Two-point spline conditions are again applied. For one relation, the following is used:

$$f_{N+1} - f_N - \frac{h_{N+1}}{2}(u_{N+1} + u_N) + \frac{h_{N+1}^2}{12}(\tau_{N+1} - \tau_N) = 0. \tag{27a}$$

Differentiating equation (27a) with respect to η yields

$$u_{N+1} - u_N - \frac{h_{N+1}}{2}(\tau_{N+1} + \tau_N) + \frac{h_{N+1}^2}{12}(l_{N+1}^\tau - l_N^\tau) = 0. \tag{27b}$$

Differentiating once more gives

$$\tau_{N+1} - \tau_N - \frac{h_{N+1}}{2}(l_{N+1}^\tau + l_N^\tau) + \frac{h_{N+1}^2}{12}(L_{N+1}^\tau - L_N^\tau) = 0.$$

Assuming the second-order terms to be negligible compared to the lower-order terms, the above equation simplifies to

$$\tau_{N+1} - \tau_N - \frac{h_{N+1}}{2}(l_{N+1}^\tau + l_N^\tau) = 0. \tag{27c}$$

Equations (27) provide the three additional conditions required for the far field. These equations are linearized and solved for the correction terms as before and then written in the following matrix form:

$$B_{N+1} Z_N + A_{N+1} Z_{N+1} = R_{N+1}. \tag{28}$$

The block tridiagonal system formed by equations (20), (25) and (28) is solved using standard lower–upper (L–U) decomposition. Partial pivoting is used within the 4 × 4 blocks to prevent the buildup of roundoff errors. A block tridiagonal solver using subroutines developed by Blottner,[4] which perform the partial pivoting, is used to solve the matrix equations for the corrections at each iteration.

The Falkner–Skan equations are obtained by setting ξ equal to zero in equations (1) and (8):

$$f_{\eta\eta\eta}+ff_{\eta\eta}+\beta(1-f_\eta^2)=0, \tag{29a}$$

$$\beta_{\eta\eta}=0. \tag{29b}$$

The boundary conditions given by equations (3) and (4), with no dependence on ξ, are used. The solution of this ordinary differential equation with splines approximating all η derivatives is used as the starting solution for the non self-similar case. The scheme is derived so that the Falkner–Skan solutions for positive and negative wall shear can also be computed.

As an initial guess for the starting solution, a fourth-order Pohlhausen-type polynomial is used to approximate $(f, u, \tau, \beta)_j$ at $\xi=0$. The Pohlhausen polynomial for the velocity is of the form

$$u=b\zeta+c\zeta^2+d\zeta^3+e\zeta^4, \tag{30}$$

where

$$\zeta=\frac{\eta}{\eta_\infty}, \quad u=u(\eta),$$

and the constants b, c, d and e can be found by applying the boundary conditions and the ordinary differential equation. The stream function and the shear can be found by integrating and differentiating equation (30) respectively. By substituting into the differential equation, equation (29a), an approximation for β is obtained in the form

$$\beta=\frac{6S\eta_\infty-12}{\eta_\infty^2}. \tag{31}$$

Once the starting solution is determined, the second streamwise station must be treated in a special way for the non-similar flow case. With only one previous streamwise step known, two-point backward differences are used to approximate the ξ derivatives. Past this station, three-point backward differences are used. The solution profile at the last calculated station is used as the first approximation at the new station.

SELF-SIMILAR FLOW SOLUTIONS

Computations for self-similar flows were performed for positive and negative wall shears. For all positive wall shears, the solutions were obtained independently for a specific shear. For the negative wall shears, the sensitivity of the method to the initial guess required calculating solutions consecutively for small steps in shear and using the previous solution as an initial guess for the next

Table I. Comparison of positive wall shear solutions for self-similar flows ($\nu\equiv$ iteration number). $\eta_\infty=6$

f_w''	REFORMULATED MECHUL FUNCTION		CEBECI & KELLER MECHUL FUNCTION		NONLINEAR EIGEN VALUE SCHEME		SMITH
	$-\beta$	ν	$-\beta$	ν	$-\beta$	ν	$-\beta$
.40032	0.050002	4	0.05025	3	0.05031	3	0.05
.31927	0.100001	4	0.10021	3	0.10017	3	0.10
.23974	0.140003	4	0.14019	3	0.14024	3	0.14
.19078	0.160008	5	0.16019	3	0.16016	3	0.16
.12864	0.180019	5	0.18020	3	0.18025	3	0.18
.05517	0.195057	6	0.19524	3	0.19528	3	0.195
0	0.198837	5	0.19917	3	0.20259	2	0.198834

Table II. Comparison of reverse-flow solutions for self-similar flows. $\eta_\infty = 9$

f''_w	REFORMULATED MECHUL FUNCTION		CEBECI & KELLER MECHUL FUNCTION		NONLINEAR EIGEN-VALUE SCHEME		STEWARTSON
	$-\beta$	ν	$-\beta$	ν	$-\beta$	ν	$-\beta$
-0.097	0.180340	3	0.18074	4	0.18143	2	0.18
-0.132	0.152116	4	0.15234	4	0.15416	6	0.15
-0.141	0.132974	4	0.13412	4	0.13545	6	—

profile. The solution process was begun for zero wall shear, $S = 0$, and the shear was decremented by 0·01 or 0·05.

Solution comparisons are made between the present reformulated scheme, the original mechul function formulation[1] and the non-linear eigenvalue scheme.[5] This method was developed by Keller and Cebeci before the mechul function scheme. The eigenvalue method solves the inverse problem by treating the unknown pressure gradient as an eigenvalue. Then, two iteration procedures, an 'inner' and an 'outer' iteration, are performed. The inner iteration solves the governing equations for a standard problem assuming β is known. The inner solution is then used with Newton's method to determine the pressure gradient parameter from the variational equations in the outer iteration procedure. The variational equations are the standard boundary layer equations differentiated with respect to β.

For positive wall shear computed with the reformulated scheme, a normal stepsize of $\Delta\eta = 0{\cdot}15$ is used with $\eta_\infty = 6$; for reverse flow solutions and separation ($S = 0$), $\Delta\eta = 0{\cdot}15$ and $\eta_\infty = 9$ are used. The criterion used for convergence is

$$\varepsilon = |\beta^{(n+1)} - \beta^{(n)}| \leqslant 10^{-8}.$$

Cebeci and Keller applied a similar convergence test to the original scheme as well as the eigenvalue scheme with $\varepsilon \leqslant 10^{-4}$.[1,5]

The results of the self-similar flow calculations with positive wall shear are given in Table I. These results are compared with those of Smith.[6] Comparison shows the values from the reformulated mechul function scheme closely approach those of Smith. All cases converged quadratically. It should be noted that the greater number of iterations required for convergence

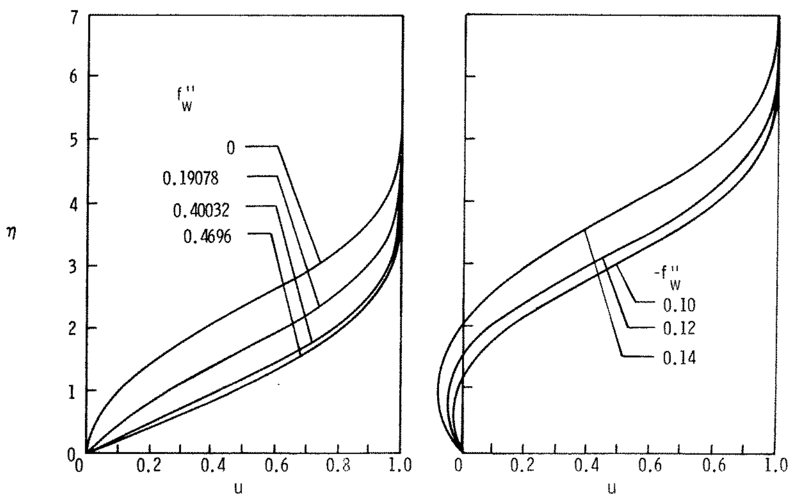

Figure 1. Boundary-layer velocity distributions for self-similar flows

with the reformulated scheme is the result of the present more severe convergence criteria.

The results of the reverse flow computations are presented in Table II. These results are compared with those of Stewartson.[7] Again, the agreement is good. Here, the number of iterations decreases appreciably with the use of consecutive calculations. Figure 1 shows examples of both positive and negative wall shear velocity profiles.

NON SELF-SIMILAR FLOW SOLUTIONS

Non self-similar flow computations were performed using two linear wall shear distributions. The first case, given by

$$S(\xi) = 0{\cdot}4696(1 - \xi)$$

Table III. Computed pressure gradient parameter β as a function of ξ for the wall shear distribution: $S(\xi) = 0{\cdot}4696\ (1 - \xi)$. $\eta_\infty = 6$

ξ	REFORMULATED MECHUL FUNCTION		CEBECI & KELLER MECHUL FUNCTION		NONLINEAR EIGEN-VALUE SCHEME		EIGENVALUE SCHEME WITH RICHARDSON EXTRAPOL.
	$-\beta$	ν	$-\beta$	ν	$-\beta$	ν	$-\beta$
0	-0.000004	4	0.00179	3	0.00179	3	0.00003
.10	0.043781	4	0.04535	3	0.04532	4	0.04383
.20	0.084248	4	0.08559	3	0.08553	5	0.09068
.30	0.121154	4	0.12231	3	0.12225	5	0.12113
.40	0.154257	4	0.15529	3	0.15522	5	0.15018
.50	0.183241	4	0.18420	3	0.18408	5	0.18308
.60	0.207657	4	0.20857	3	0.20845	5	0.20747
.70	0.226840	4	0.22685	3	0.22761	4	0.22664
.80	0.239734	5	0.22378	3	0.24041	4	0.23940
.90	0.251269	8	0.16835	3	0.24479	3	0.24376

Table IV. Compared of β as a function of ξ with varying η_∞, for the wall shear distribution: $S(\xi) = 0{\cdot}4696(1 - \xi)$

ξ	MECHUL FUNCTION WITH $\eta_\infty = 6$		MECHUL FUNCTION WITH $\eta_\infty = 9$	
	$-\beta$	ν	$-\beta$	ν
0	-0.000004	4	-0.000004	4
.10	0.043781	4	0.043780	4
.20	0.084248	4	0.084247	4
.30	0.121154	4	0.121152	4
.40	0.154257	4	0.154254	4
.50	0.183241	4	0.183234	4
.60	0.207657	4	0.207637	4
.70	0.226840	4	0.226777	4
.80	0.239734	5	0.239550	4
.90	0.251269	8	0.243963	4
1.00	—	—	0.235625	5
1.10	—	—	0.207213	7

Table V. Computed pressure gradient β as a function of ξ for the wall shear distribution: $S(\xi) = 1{\cdot}23259\,(1-\xi)$. $\eta_\infty = 6$

ξ	REFORMULATED MECHUL FUNCTION		NONLINEAR EIGENVALUE SCHEME	EIGENVALUE SCHEME W/ RICHARDSON EXTRAPOLATION
	β	ν	β	β
0	0.999993	4	0.98862	0.99938
.10	0.760840	4	0.75308	0.76053
.20	0.542190	4	0.53733	0.54213
.30	0.344378	4	0.34164	0.34445
.40	0.167607	4	0.16634	0.16772
.50	0.012386	4	0.01203	0.01248
.60	-0.120291	4	0.12017	-0.12017
.70	-0.228585	4	0.22837	-0.22844
.80	-0.308985	4	-0.30877	-0.30856
.90	-0.352992	7	-0.35302	-0.35232
.95	-0.366964	10	-0.45517	-0.35427

has a zero pressure gradient at $\xi = 0$, indicating a flat plate flow. The second case, given by

$$S(\xi) = 1{\cdot}23259(1-\xi)$$

has a pressure gradient parameter of unity at $\xi = 0$, indicating a stagnation point. Both cases approach zero shear at $\xi = 1$, yielding separation. Both cases were computed with values of $\eta_\infty = 6$ and $\eta_\infty = 9$. For all non self-similar computations, a streamwise stepsize of $\Delta\xi = 0{\cdot}05$ was used, with $\Delta\eta = 0{\cdot}25$. The convergence criterion applied is identical to that used for the self-similar cases.

A comparison of results for the first case is given in Table III and IV. Table III compares the results with those of References 1 and 5. The agreement with the Richardson extrapolation results obtained from the eigenvalue scheme values is quite good. The numerical instability experienced by the original formulation does not appear in the present method. It was found that with $\eta_\infty = 6$, the solution would not converge at streamwise stations near separation. With $\eta_\infty = 9$, the solution was

Table VI. Comparision of β as a function of ξ with varying η_∞, for the wall shear distribution: $S = 1{\cdot}23259\,(1-\xi)$

ξ	MECHUL FUNCTION WITH $\eta_\infty = 6$		MECHUL FUNCTION WITH $\eta_\infty = 9$	
	β	ν	β	ν
0	0.999993	4	0.999993	4
.10	0.760840	4	0.760840	4
.20	0.542190	4	0.542190	4
.30	0.344378	4	0.344378	4
.40	0.167607	4	0.167607	4
.50	0.012386	4	0.012386	4
.60	-0.120291	4	-0.120289	4
.70	-0.228585	4	-0.228577	4
.80	-0.308985	4	-0.308911	4
.90	-0.352992	7	-0.353486	4
1.00	—		-0.339311	5

marched through the separation point at $\xi = 1$. Once past separation, however, the scheme quickly became unstable and the solution failed to converge. Table IV compares present results for the first case with $\eta_\infty = 6$ and $\eta_\infty = 9$.

A comparison of results for the second case is given in Tables V and VI. In Table V, the results are compared with results from Reference 5, since no results are available for this case with the original mechul function formulation. The agreement with the Richardson extrapolation of Reference 5 is again very good. As with case one, the results could be obtained through separation only with $\eta_\infty = 9$. Table VI compares the results for the two values of η_∞.

For all numerical cases presented, the normal stepsize, $\Delta\eta$, is kept constant in order to compare with References 1 and 5. Non self-similar cases were also computed, using a geometric progression for η, with the stepsize ratio $\sigma_j = 1{\cdot}1$. Using this geometric progression, both non self-similar cases proceeded through separation with $\eta_\infty = 6$. The results obtained are identical with those produced with a constant stepsize and $\eta_\infty = 9$. Using the geometric progression also allowed the number of normal steps to be halved without decreasing solution accuracy.

CONCLUSION

An accurate, stable and efficient method has been developed to determine the pressure gradient distribution on a body surface in a laminar boundary layer with wall shear specified. The reformulated mechul function scheme presented here does not exhibit the numerical instability experienced with non self-similar type flows in Reference 1. The method is fully implicit and is applicable to Falkner–Skan as well as non self-similar flows. Reverse flow self-similar solutions have also been computed but are found to be very sensitive to the initial solution guess.

The reformulated scheme uses fourth-order splines to approximate derivatives in the normal direction and three-point backward differences for the streamwise derivatives to provide the necessary stability as the solution is marched downstream. The buildup of roundoff errors is prevented by the use of partial pivoting within the 4×4 blocks of the linearized system of spline–finite difference equations. Accurate results have been found with the scheme for both self-similar and non self-similar cases, all of which exhibit quadratic convergence. The non self-similar calculations were observed to proceed slightly beyond separation with constant stepsize in η and $\eta_\infty = 9$, or with a geometric progression and $\eta_\infty = 6$.

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